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# ON THE DIFFERENTIATION OF DEFINITE INTEGRALS

BY WM. F. OSGOOD

THE object of this paper is to give a simpler proof of the theorem that

$$(A) \quad \frac{d}{da} \int_a^b f(x, a) dx = \int_a^b \frac{\partial f}{\partial a} dx + f(b, a) \frac{db}{da} - f(a, a) \frac{da}{da}$$

than those that are current.

1. **The Common Proofs.** The theorem is usually proven by writing

$$\phi(a) = \int_a^b f(x, a) dx,$$

forming the difference :

$$\begin{aligned} \phi(a + \Delta a) - \phi(a) &= \int_{a+\Delta a}^{b+\Delta b} f(x, a + \Delta a) dx - \int_a^b f(x, a) dx \\ &= \left( \int_{a+\Delta a}^a + \int_a^b + \int_b^{b+\Delta b} \right) f(x, a + \Delta a) dx - \int_a^b f(x, a) dx \\ &= \int_a^b [f(x, a + \Delta a) - f(x, a)] dx + \int_b^{b+\Delta b} f(x, a + \Delta a) dx \\ &\quad - \int_a^{a+\Delta a} f(x, a + \Delta a) dx, \end{aligned}$$

and applying the theorems of mean value to these last three integrals; suitable assumptions being made about the continuity of the functions that enter. Cf., for example, Goursat-Hedrick, *Mathematical Analysis*, vol. I, §97.

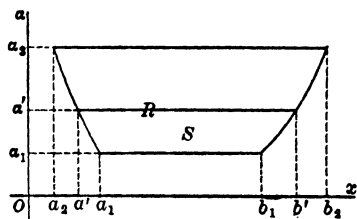
Another proof consists in changing the variable of integration so that the new limits of integration become constant:

$$t = \frac{x - a}{b - a}, \quad x = (b - a)t + a,$$

$$\int_a^b f(x, a) dx = (b - a) \int_0^1 f(x, a) dt,$$

this latter case having been treated previously. Cf. Picard, *Traité d'analyse*, vol. I, chap. I, §18.

**2. Critique of these Proofs.** We will first state the theorem in detail in its simplest and most useful form. Let  $R$  be a region of the  $(x, a)$ -plane bounded by the right lines



$$a = a_1, \quad a = a_2, \quad (a_1 < a_2),$$

and the curves

$$a = \psi(a), \quad b = \omega(a),$$

where each of the functions  $\psi(a)$  and  $\omega(a)$  shall be continuous, together with its first derivative, throughout the interval  $a_1 \leq a \leq a_2$ , and where

$$\psi(a) < \omega(a).$$

The region  $R$  shall include its boundary.

The function  $f(x, a)$  shall be continuous at all points of  $R$ . Its partial derivative

$$\frac{\partial f}{\partial a} = f_a(x, a)$$

shall exist and be continuous at all interior points of  $R$ , and this function,  $f_a(x, a)$ , shall remain finite throughout the interior of  $R$ .

*Under the above conditions the integral*

$$\int_a^b f(x, a) dx$$

*defines a function of  $a$  having a derivative given by the above formula, (A).*

Turning now to the first of the proofs cited in §1 we see that if, in the neighborhood of a point  $a = a'$  of the interval  $a_1 \leq a \leq a_2$ , the functions  $\psi(a)$  and  $\omega(a)$  are both monotonic, the above transformation of the difference of the integrals, or a suitable modification of this transformation, is legitimate and the proof is sound. But there are several cases to consider when  $\Delta a$  and  $\Delta b$  change sign with  $\Delta a$ .

If, in particular, the functions  $f(x, a)$ ,  $f_a(x, a)$  are continuous throughout a larger region  $R'$  lying between the same parallels  $a = a_1$  and  $a = a_2$  and containing all the interior and boundary points of  $R$  situated between these

lines in its interior, the single transformation of the difference given above will hold for both positive and negative values of  $\Delta a$ . But the theorem thus restricted is too narrow for all the ordinary applications of practice.

There are, then, even when  $\psi(a)$  and  $\omega(a)$  are both monotonic, several cases of the transformation of the difference of the above integrals to be considered.

If, however, one of the functions  $\psi(a)$ ,  $\omega(a)$  is not monotonic; for example, if

$$\begin{aligned}\omega(a) &= (a - a')^2 \sin \frac{1}{a - a'}, & a \neq a', \\ \omega(a') &= 0,\end{aligned}$$

there is trouble. The theorem is still true, but a special  $\epsilon$ -proof is necessary.

The second proof is not open to this objection, but it turns out that it is necessary to assume the existence of the partial derivative of  $f(x, a)$  with respect to  $x$ , and thus the proof does not apply to the theorem stated in the above generality.\*

**3. A New Proof.** We can obtain a simple proof of the theorem as follows. Consider the double integral

$$(1) \quad I = \int_S f_*(x, a) dS,$$

extended over the region

$$S: \quad a \leq x \leq b, \quad a_1 \leq a \leq a',$$

when  $a_1 < a' \leq a_2$ . We can first evaluate  $I$  by means of the iterated integral

$$(2) \quad I = \int_{a_1}^{a'} da \int_a^b f_*(x, a) dx.$$

Secondly, we can evaluate  $I$  by means of Green's Theorem:

$$\int_S \int f_*(x, a) dS = - \int_C f(x, a) dx.$$

\* By means of this transformation, however, a simple proof can be given that

$$\int_a^b f(x, a) dx$$

is a continuous function of  $a$  if  $f(x, a)$  is continuous in  $R$ , and  $a, b$  are continuous functions of  $a$ . The existence of a derivative with respect to  $x$  is here unnecessary.

Hence

$$(3) \quad I = - \int_{a_1}^{b_1} f(x, a_1) dx - \int_{a_1}^{a'} f(b, a) \frac{db}{da} da \\ + \int_{a'}^{b'} f(x, a') dx + \int_{a_1}^{a'} f(a, a) \frac{da}{da} da.$$

Equating the two expressions (2) and (3) for  $I$ , dropping the accent against the  $a$ , and differentiating the equation thus resulting with respect to  $a$ , we obtain the theorem contained in the formula ( $A$ ).

Instead of using the region  $S$  between the parallels  $a = a_1$  and  $a = a'$ , we might have chosen an arbitrary value  $a_3 \neq a'$ . The two evaluations of  $I$  would have yielded the same final equation, the subscript 1 being merely replaced by 3. Thus, in particular, if  $a_3 = a_2$ , the excluded value  $a' = a_1$  no longer presents an exception.

**4. Generalizations.** Corresponding to more general forms of Green's Theorem we obtain the theorem under consideration with less restrictive hypotheses. Thus if  $f_*(x, a)$  is continuous within  $R$ , but does not remain finite on the boundary, the function  $f(x, a)$  still being assumed continuous in  $R$ , and if the surface integral (1) converges when extended over any region

$$S': \quad a \leq x \leq b, \quad \bar{a}_1 \leq a \leq \bar{a}_2,$$

where  $a_1 < \bar{a}_1 < \bar{a}_2 < a_2$ , and if, moreover, the first integral to be evaluated in (2), namely:

$$\int_a^b f_*(x, a) dx,$$

converges uniformly in the interval  $\bar{a}_1 \leq a \leq \bar{a}_2$ , the above proof will hold for all values of  $a'$  such that  $a_1 < a' < a_2$ .

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